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The Oscillation of Functions of an Orthogonal Set.*

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1. Introductory.

The sets of orthogonal functions which occur in mathematical physics have, in general, the property that each changes sign in the interior of the interval on which they are orthogonal once more than its predecessor. So universal is this property that such sets are frequently referred to as sets of "oscillating functions." The question arises, is this property of oscillation inherent in that of orthogonality? That it is not, is evidenced by a simple example. If the first function does not vanish, it is clear that the second must change signs, but the example shows that it does not follow that the third must change signs twice. Thus let $\phi_0(x)=1$, while $\phi_1(x)$ is defined on the interval $(0,1)$ as follows:

$$\begin{aligned}\text{For } 0 \leq x \leq \tfrac{1}{3}, \quad \phi_1(x) &= 27x-8, \\ \tfrac{1}{3} \leq x \leq \tfrac{1}{2}, \quad \phi_1(x) &= 18x-5, \\ \tfrac{1}{2} \leq x \leq \tfrac{2}{3}, \quad \phi_1(x) &= -18x+13, \\ \tfrac{2}{3} \leq x \leq 1, \quad \phi_1(x) &= 1.\end{aligned}$$

Let $\phi_2(x)=\phi_1(1-x)$. Then the three functions are orthogonal on the interval $(0,1)$, while the second two change signs but once each.

If $\phi_2(x)$ changes signs but once, say at $x=a$, the function $\phi_0(a)\phi_1(x)-\phi_1(a)\phi_0(x)$, which is orthogonal to $\phi_2(x)$, can not have $x=a$, where it vanishes, as the only point where it changes signs, since two orthogonal functions must be of the same sign in part of the interval and of opposite signs in part. Hence, if we make the supposition $\phi_0(x_0)\phi_1(x_1)-\phi_0(x_1)\phi_1(x_0) > 0$ for $0 < x_0 < x_1 < 1$, $\phi_2(x)$ must change signs twice in the interior of the interval.

These considerations suggest the determinant condition of the next paragraph. It is interesting to note that it is essentially the condition that a function $c_0\phi_0(x)+c_1\phi_1(x)+\dots+c_n\phi_n(x)$ can be found which will coincide with a given function $f(x)$ at any $n+1$ interior points of the interval, the difference being merely the substitution of a definite sign for the being different from

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zero. Also it should be noted that it is a condition satisfied by the more common orthogonal sets, like the sines, $\sin \pi x$, $\sin 2\pi x$, $\sin 3\pi x$, . . . ; the Legendre polynomials; and the Bessel functions, $\sqrt{x}J_0(\alpha_1 x)$, $\sqrt{x}J_0(\alpha_2 x)$, $\sqrt{x}J_0(\alpha_3 x)$, . . . , where $\alpha_1, \alpha_2, \alpha_3, \dots$ are the roots of $J_0(x) = 0$. The subsequent paragraphs will be devoted to deriving some oscillation and other properties from the suggested conditions.

2. Oscillation Properties.

Let $\phi_0(x), \phi_1(x), \dots, \phi_n(x), \dots$ be a set of real continuous functions, normed and orthogonal, on the interval $(0, 1)$. Let the determinants

$$D(x_0, x_1, x_2, \dots, x_n) = \begin{vmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_n(x_1) \\ \dots & \dots & \dots & \dots \\ \phi_0(x_n) & \phi_1(x_n) & \dots & \phi_n(x_n) \end{vmatrix}$$

be positive for any $x_0, x_1, \dots, x_n$ in the interior of the interval $(0, 1)$ and in ascending order of magnitude, this condition to hold for $n=0, 1, 2, \dots$, $D_0(x_0)$ being understood as $\phi_0(x_0)$. Let $\Phi_{m,n}(x)$ denote the function $c_m\phi_m(x) + c_{m+1}\phi_{m+1}(x) + \dots + c_n\phi_n(x)$.

The following theorems are either evident or are made so by the brief proofs indicated:

(1) *Given $n+1$ distinct points $x_0, x_1, \dots, x_n$ in the interior of $(0, 1)$, $c_0, c_1, \dots, c_n$ may be so chosen that $\Phi_{0,n}(x)$ will take on any given values at these points.*

(2) *$\Phi_{0,n}(x)$ can not vanish at $n+1$ distinct points in the interior of $(0, 1)$ without vanishing identically.*

I. *If $\Phi_{0,n}(x)$ vanishes at n distinct points, it changes sign at each. For it may be written in the form*

$$\Phi_{0,n}(x) = kD(x, x_0, x_1, \dots, x_{n-1}),$$

where k is a constant other than zero, and $D(x, x_0, x_1, \dots, x_{n-1})$ is positive or negative according as the number of inversions in the sequence $x, x_0, x_1, \dots, x_{n-1}$ is even or odd.

(3) *$\phi_n(x)$ can not vanish more than n times. For otherwise $D(x_0, x_1, \dots, x_n)$ could vanish, the arguments being all different.*

II. *Every continuous function $\psi(x)$ orthogonal to $\phi_0(x), \phi_1(x), \dots, \phi_n(x)$ on the interval $(0, 1)$ changes signs at least $n+1$ times. For, if $\psi(x)$ changed signs at $x_0, x_1, \dots, x_k$ only, k being less than n , we could determine a function $\Phi_{0,k+1}(x)$ which would vanish at these points, and which at one other*

into $n-1$ parts in each of which $F_0(x)$ does not change signs. As $F_0(a) \neq 0$, the point a need not be one of the set $a_1, a_2, \dots, a_{n-2}$. Suppose it comes between a_i and a_{i+1} . The function $\Phi(x) = c_1 F_1(x) + c_2 F_2(x) + \dots + c_{n-2} F_{n-2}(x) + c_{n-1} \phi_{n-1}(x) + c_n \phi_n(x)$ will be orthogonal to $F_0(x)$ for every choice of $c_1, c_2, \dots, c_n$. We chose these constants so that

$$\begin{aligned} \int_0^{a_1} F_0(x) \Phi(x) dx &= \int_{a_1}^{a_2} F_0(x) \Phi(x) dx = \dots = \int_{a_i}^a F_0(x) \Phi(x) dx \\ &= \int_a^{a_{i+1}} F_0(x) \Phi(x) dx = \dots = \int_{a_{n-2}}^1 F_0(x) \Phi(x) dx = 0. \end{aligned}$$

This gives $n-1$ homogeneous linear equations in the n constants c_i , so that the conditions are compatible, the c_i not all vanishing. But as $\int_0^1 F_0(x) \Phi(x) dx = 0$, it follows also that $\int_{a_{n-2}}^1 F_0(x) \Phi(x) dx = 0$. But $F_0(x)$ keeps its sign in each interval of integration; hence $\Phi(x)$ must change signs in the interior of each, or n times. But it also vanishes at one end point a , since each term vanishes there. Thus we are led to the conclusion that $\Phi(x)$, which is a function $\Phi_{0,n}(x)$, vanishes $n+1$ times, and this contradicts theorem V. Hence, $\phi_{n-1}(x)$ and $\phi_n(x)$ can not have a common root.

VII. $\phi_{n-1}(x)$ changes sign between two successive roots of $\phi_n(x)$. Suppose this were not the case. Let a and b be two successive roots of $\phi_n(x)$. We may then, without loss of generality, suppose $\phi_n(x)$ positive in the open interval (a, b) , and $\phi_{n-1}(x)$ positive in the closed interval $[a, b]$, for it will not vanish at a or b , by VI. Consider the function $\Phi(x, t) = \phi_{n-1}(x) - t\phi_n(x)$. For $t=0$ it is positive in the closed interval $a \leq x \leq b$, and the same is true for sufficiently small positive values of t . For some finite positive value of t , say T , $\Phi(x, T)$ will have some negative values in the same interval. There is therefore a value of t , say τ , such that for every positive ε less than τ $\Phi(x, \tau - \varepsilon) > 0$ throughout the interval $a \leq x \leq b$, whereas $\Phi(x, \tau + \varepsilon)$ has negative values in this interval. The function $\Phi(x, \tau)$ can not be negative in the interval (a, b) , since $\Phi(x, t)$ is continuous. For the same reason its minimum can not be positive. Hence in the closed interval $[a, b]$, $\Phi(x, \tau)$ is never negative, but vanishes at one or more points, which, by VI, must be in the interior. But as $\Phi(x, \tau)$ is a function $\Phi_{n-1,n}(x)$, it must change signs at least $n-1$ times, by V. These sign changes must occur outside (a, b) , so that altogether $\Phi(x, \tau)$ vanishes n times, without, however, changing signs at all n points. This contradicts theorem I, so that $\phi_{n-1}(x)$ can not keep its sign between two successive roots of $\phi_n(x)$.

(4) *The roots of $\phi_n(x)$ and $\phi_{n-1}(x)$ occur alternately.* This is an immediate corollary of VII, since $\phi_n(x)$ has exactly n roots and $\phi_{n-1}(x)$ exactly $n-1$.

4. *Remarks.*

Essentially the conditions on a set of functions which have been employed are (1) orthogonality on some interval I , and (2), $D(x_0, x_1, \dots, x_n) > 0$ when $x_0, x_1, \dots, x_n$ are in the interior of the interval I and in ascending order of magnitude, for all n . The functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \dots$ do not satisfy the second condition for every n , the interval I being $(0, 2\pi)$. They do, however, for every even n . This suggests a modification of the condition in which we merely assume an infinite sequence of integers $n_1, n_2, n_3, \dots$ for which the $D(x_0, x_1, \dots, x_n) > 0$. Theorems of the same type as those obtained follow, with appropriate modification. Thus, in place of V we would have

V'. *If $n_i < m$ and $n \leq n_j$, then for any fixed $c_m, c_{m+1}, \dots, c_n$, not all zero, $\Phi_{m,n}(x) = c_m \phi_m(x) + c_{m+1} \phi_{m+1}(x) + \dots + c_n \phi_n(x)$ changes sign at least $n_i + 1$ times and at most n_j times.*

The closure property of theorem IV is weaker than is desirable. All we can conclude from it, in the matter of the determination of a function by its generalized Fourier constants, is that the function is determined but for an additive function which is zero at an infinite number of points.

It is further desirable to make the connection of the above theory with that of integral equations. The chief matter of interest would be to find a condition on the kernel of the integral equation which corresponds to the determinant condition on its characteristic functions. I hope to have some results along these lines in the near future.

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